The Collatz tree as a Hilbert hotel

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The Collatz tree as a Hilbert hotel: a proof of the $3x + 1$ conjecture

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Abstract

The yet unproven Collatz conjecture maintains that repeatedly connecting even numbers n to $n/2$, and odd n to $3n+1$, connects all natural numbers by a unique *root path* to the *Collatz tree* with 1 as its root. The Collatz tree proves to be a *Hilbert hotel*. Numbers divisible by 2 or 3 depart. An infinite binary tree remains with one *upward* and one *rightward* child per number. Rightward numbers, and infinitely many generations of their upward descendants, each with a well-defined root path, depart thereafter. The Collatz tree is a Hilbert hotel because even higher upward descendants keep descending to all unoccupied nodes. The *density of already departed numbers* comes nevertheless arbitrarily close to 100% of the natural numbers. The latter proves the Collatz conjecture.

1 The Collatz tree

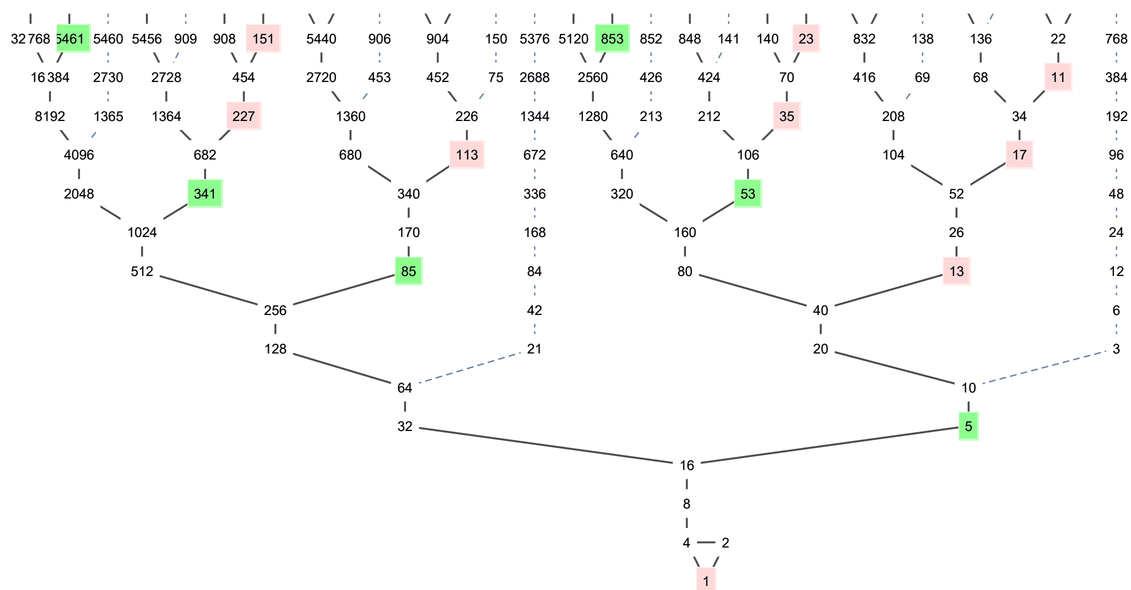
The Collatz conjuncture maintains that the Collatz function $C(n) = n/2$ if n is even, but $C(n) = 3n+1$ if n is odd, reaches 1 for all natural numbers n, in a finite *root path* of iterations, each of them resulting in either expansion or contraction.

$8 \to 4 \to 2 \to 1$ $(\to 4 \to 2 \to 1 \to \dots)$
$9 \to 28 \to 14 \to 7 \to 22 \to 11 \to 34 \to 17 \to 52 \to 26 \to 13 \to 40 \to 20 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1$
$10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1$

This one-sentence conjecture became famous after it had been detected that its proof would not be as easy. “The problem appeared to be representative of a large class of problems concerning the behavior under iteration of maps that are expanding on part of their domain and contracting on another part of their domain.” (*1*). In the words of Paul Erdős (1913-1996): ”hopeless, absolutely hopeless” (*1*). Numerical computations verified the conjecture for all numbers below 2^{68}(*2*). In the paper on his recent proof that the conjecture holds for "almost all" numbers, Terence Tao reckons nevertheless that "a full resolution of the conjecture remains well beyond current methods" (*3*).

If the Collatz conjecture would hold, then each number n from the set of natural numbers $\mathbb{N}=\{1,2,3,\dots\}$ would also belong to the node set $N(T_{\mathbb{C}})$ of one *Collatz tree* $T_{\mathbb{C}}$, as would be evidenced by a unique *root path* of Collatz iterations between n itself and the tree's trivial cyclic root, denoted as $\Omega_C=\{1,4,2\}$, No *isolated trajectory* would exist, neither a *divergent trajectory* from n to infinity, nor a *nontrivial cycle* from any $n>4$ back to its origin n (*1*). To be proven is that $N\,(T_C)$=$\mathbb{N}$. Lothar Collatz (1910-1990) hoped to prove the conjecture "using the fact that one can picture a number theoretic functions $f(n)$ with a directed graph" with for each iteration "an arrow from n to $f(n)$", thereby strengthening the "connections between elementary number theory and elementary graph theory" (*4*). In Figure 1 the lowest part of the Collatz tree is pictured, already annotated with node subsets to consider the Collatz tree as a *Hilbert hotel*.

Figure 1: The Collatz tree



Legend: Dashed orbits indicate numbers divisible by 3. Numbers divisible by 2 and/or 3 (subset S_{-1}) will depart preliminarily. The distinction between red numbers (here subset S_0) that depart thereafter and green numbers (here subsets $S_{\geq 1}$) recurs in all Figures.

2 A Hilbert hotel

A Hilbert hotel is an infinite hotel that never gets full upon the arrival of – an infinite number of sets of – new guests because all hotel guests can move to higher numbered rooms (*5*). A Hilbert hotel with the architecture of a Collatz tree enables an orderly departure of infinitely many sets of infinitely many numbers, because their children who were sitting one branch higher take their place (to be shown in Figure 3 below). The Hilbert hotel analogue revitalizes an hitherto deficient approach spelled out by Chamberland to shift the burden of proof by finding *ever less dense* node subsets $S \subset \mathbb{N}$ of the Collatz tree such that proving the conjecture on S implies it is true on $\mathbb{N}$ (6).

Here we will label the ever less dense remaining node subsets as $S_{\geq 0},\ S_{\geq 1}, S_{\geq 2}, S_{\geq 3},\ \ldots, S_{\geq j+1}, \ldots$, and the corresponding just departed subsets as $S_{-1},\ S_0,\ S_1, S_2, \ldots, S_j, \ldots$. After all numbers from the increasingly less dense node subsets $S_{\geq 0},\ S_{\geq 1}, S_{\geq 2}, S_{\geq 3},\ \ldots, S_{\geq j+1}, \ldots$ have descended one node compared to their previous position, they nevertheless occupy all nodes in the reconnected *binary* successor trees $T_{\geq 0},\ T_{\geq 1}, T_{\geq 2}, T_{\geq 3},\ \ldots, T_{\geq j+1}, \ldots$. In each successor tree, the node of each departed number m will be taken by its unique *upward* child n. The proof of the conjecture that $N(T_c) = \mathbb{N}$ now rests on the proof of two theorems.

First, a proof that the burden of proof can be shifted indeed. *If* the existence of a root path can be established for a not yet departed number m in a successor tree $T_{\geq j}$, then a root path must have existed in tree $T_{\geq 0}$ for its uniquely identifiable ancestor who departed from the tree (theorem 1). Next a proof that the *natural density* of the already departed number sets $S_{-1}, S_0,\ S_1, S_2, \ldots S_j$, also denoted as the sets $S_{\geq -1}$ comes arbitrarily close to 100% of the natural numbers $\mathbb{N}$, which implies that the density of $S_{\geq j}$, on which a proof has to be delivered, has come arbitrarily close to 0% (lemma 2).

Let's illustrate the purport of these proofs with straightforward proofs for the preliminary departing subset S_{-1} of numbers divisible by 2 or 3. These are also removed beforehand in parts of the research literature (7). They remained uncoloured in Figure 1. Any number divisible by both 2 and 3 – depicted on dashed orbits – cannot be reached by $3n+1$ itself, while iterations of $n/2$ lead to a number that is divisible by 3 only, after which $3x+1$ leads to an even number that is not divisible by 3. For example, $852 = 2^2 \cdot 3 \cdot 71$, pictured in the middle of the highest level in Figure 1, leads through two $n/2$ iterations to $213 = 3 \cdot 71$, and through $3x+1$ to $640 = 2^7 \cdot 5$. *If* it could be proven that even numbers n not divisible by 3 would have a root path, *then* the conjecture would hold also on numbers divisible by 3. Even numbers not divisible by 3, like 640, merely *intermediate* in between numbers $(2^p n - 1)/3$ and numbers $n/2^q$ from subset $S_{\geq 0}$ of numbers neither divisible by 2 nor by 3. Number $640 = 2^7 \cdot 5$ intermediates between, among others, $853 = (2^2 \cdot 640 - 1)/3$ on the one hand and $5 = 640/2^7$ on the other. *If* all numbers in $S_{\geq 0}$ would have a root path, *then* the conjecture would hold on S_{-1} also.

Obviously the *natural density* of subset S_{-1}, denoted with Greek rho as $\rho(S_{-1})$, amounts to $\rho(S_{-1}) = 4/6$, since 4 out of 6 successive numbers are divisible by either 2 or 3. They comprise four *congruence classes* out of six congruence classes with a modulus, or periodicity, of 6, denoted with Greek nu as $\nu = 6$, i.e. the set $S_{-1} = \{0, 2, 3, 4\} \equiv n \bmod 6$, for $n \in \mathbb{N}$. Each of the four congruence classes represents an *arithmetic progression*. Class $2 \equiv n \bmod 6$ is equivalent to the arithmetic progression $6i + 2 = \{2, 8, 14, \ldots\}$ for $i \in \mathbb{N}^0$ with $\mathbb{N}^0 = \{0, \mathbb{N}\}$. In the density calculations below $\#(S_{-1}|\nu) = 4$ denotes the *cardinality per period*.

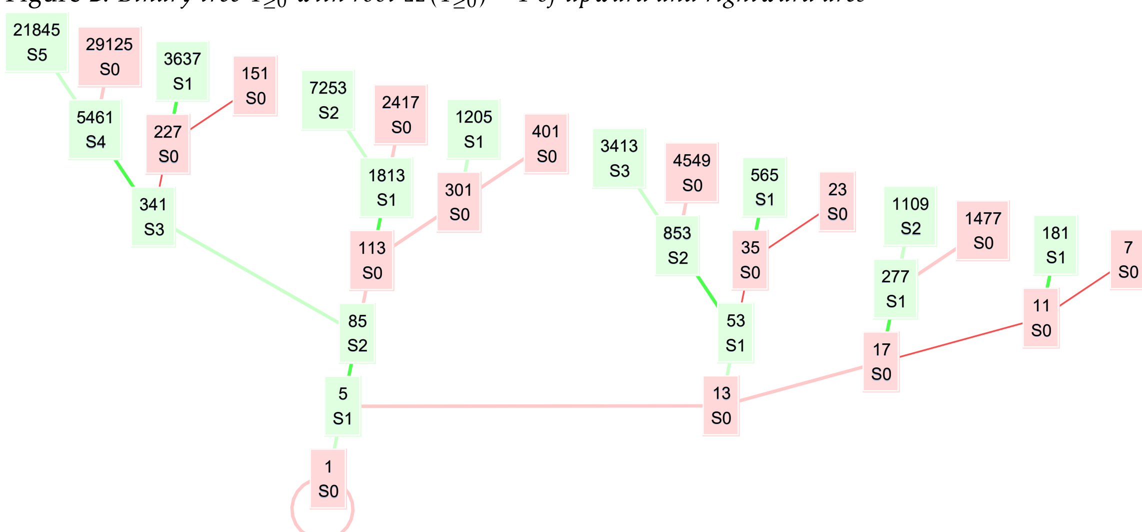
3 From the Syracuse tree to a binary tree

Our major innovation is to accommodate the remaining numbers $S_{\geq 0} = (\{1, 5\} \equiv n \bmod 6)$ in a Hilbert hotel with the architecture of a *binary* Collatz tree, labelled $T_{\geq 0}$, that enables further departures of number sets $S_0,\ S_1, S_3, \ldots$. Let's first discuss the *Syracuse tree* of odd numbers that gained prominence in the research literature (1, 3). The *Syracuse tree* function *Syr*(n) connects any *odd* number n directly to the next *odd* number obtained by iterations of the Collatz function. The *inverse* Syracuse function therefore connects each odd node *n in parallel* to infinitely many children $Syr^{-1}(n) = (2^p n - 1)/3$, granted by *infinitely many* powers $p \in \mathbb{N}$ for which both n and $(2^p n - 1)/3$

are odd. For example, $Syr^{-1}(1)$ connects its argument 1 *in parallel* to $1 = \left(2^2 1 - 1\right)/3$, thus to itself, and moreover to $5 = \left(2^4 1 - 1\right)/3$, $85 = \left(2^8 1 - 1\right)/3$, $341 = \left(2^{10} 1 - 1\right)/3$, $5461 = \left(2^{14} 1 - 1\right)/3$, and so on, which are all shown in green in Figure 1. Infinitely many $Syr^{-1}(n)$ children to each node n preclude *natural density* calculations *per finite period*, but a measure of *logarithmic density* enabled Tao nevertheless to prove the Collatz conjecture on the Syracuse tree for "almost all" numbers (3).

The binary tree $T_{\geq 0}$ in contrast, of which the lowest part is depicted in Figure 2, connects $1 \to 5 \to 85 \to 341 \to 5461 \to \ldots$ in a *serial* upward orbit, based on the following three adaptations of the inverse Syracuse function Syr^{-1}. First, both odd arguments n and odd outputs $Syr^{-1}(n)$ are also required to be non-divisible by 3. Next, n's infinite offspring $Syr^{-1}(n)$ is limited to the minimum value of p, thus to its first-born child, labeled as its *rightward*, red colored, child $R_0(n)$ for which p in $\operatorname{argmin}_p \left(2^p n - 1\right)/3$ is minimal. Third, the first born child to n's even, already departed, parent becomes n's *upward*, green colored, child $U(n)$, for which p in $\operatorname{argmin}_p \left(2^p (3n+1) - 1\right)/3$ is minimal. A value of p is the minimum argument if all lower values of p result either in fractions or in numbers divisible by 3.

Figure 2: *Binary tree $T_{\geq 0}$ with root $\Omega(T_{\geq 0}) = 1$ of upward and rightward arcs*



Legend: Each number in tree $T_{\geq 0}$ is labeled with the subset to which it belongs. Upward functions and upward nodes in subsets $S_{\geq 1}$ are colored green. Rightward functions and rightward nodes in subset S_0 are colored red because they are the first to leave the tree. Rightward contractions, for $\{11, 17\} \equiv n \bmod 18$, are shown in bright red, and large upward extractions, for $5 \equiv n \bmod 6$, in bright green.

Definitions 1 and 2 of functions U and R_0, both based on argument minimization, give fixed minimum powers p that map arguments from infinite argument congruence classes, respectively modulo 6 and modulo 18, to *disjoint* infinite output congruence classes with a least common modulus of 96. Definition 3 of the binary tree $T_{\geq 0}$ uses the definitions of U and R_0 to partition its node set $S_{\geq 0}$ into the remaining subsets $S_1, S_2, S_3, \ldots,$ – colored green in Figure 2, denoted as $S_{\geq 1}$ –and a departing subset S_0 of rightward numbers – colored red in Figure 2. To allow also for the existence of numbers on hypothetical isolated trajectories, definitions 1 and 2 start from potentially broader subsets $\mathbb{N}^{UR} \supseteq S_{\geq 0}$, $\mathbb{N}^{U} \supseteq S_{\geq 1,}$ and $\mathbb{N}^{R} \supseteq S_0$ in their domain-to-codomain specifications $U : \mathbb{N}^{UR} \to \mathbb{N}^{U}$, respectively $R_0 :\ \mathbb{N}^{UR} \to \mathbb{N}^{R}$. The burden to prove $N(T_C) = \mathbb{N}$ is shifted to the proof of $S_{\geq 0} = \mathbb{N}^{UR}$.

Definition 1, upward function. $U :\ \ \mathbb{N}^{UR} \to \mathbb{N}^{U};\ U(n) = \operatorname{argmin}_p (2^p(3n+1)-1)/3$

$$\begin{array}{ccccc} \overrightarrow{a} \quad \overrightarrow{p}(a) \quad U(n) & & n = 6i + a & & \mathbb{N}^{U} \\ = \left\{ \begin{array}{c} \text{if } 1 \equiv n \bmod 6 : p = 2 \hookrightarrow\ 4n+1 \\ \text{if } 5 \equiv n \bmod 6 : p = 4 \hookrightarrow 16n+5 \end{array} \right\}; & & \left\{ \begin{array}{c} 1,7,13,19,\ \ldots \\ 5,11,17,23\ldots \end{array} \right\} & \to & \left\{ \begin{array}{r} 5,29,53,77 \\ 85 \end{array} \right\} \equiv U(n) \bmod 96, \end{array}$$

with $U^2(n) = 64n + 21$, periodicity expansions $\theta_{U1} = 96/6 = 16$ and $\theta_{U2} = 64$,
and a upward heap vector $\overrightarrow{h}_U(a) = [4, 1]$.

Definition 2, rightward function. $R_0 :\ \ \mathbb{N}^{UR} \to \mathbb{N}^{R};\ \ R(n) = \operatorname{argmin}_p (2^p n - 1)/3$

$$\begin{array}{ccccc} \overrightarrow{a} \quad \overrightarrow{p}(a) \quad R(n) & & n = 18\,i + a & & \mathbb{N}^{R} \\ = \left\{ \begin{array}{c} \text{if } 1 \equiv n \bmod 18 :\ p = 2 \hookrightarrow (4n-1)/3 \\ \text{if } 5 \equiv n \bmod 18 :\ p = 3 \hookrightarrow (8n-1)/3 \\ \text{if } 7 \equiv n \bmod 18 :\ p = 4 \hookrightarrow (16n-1)/3 \\ \text{if } 11 \equiv n \bmod 18 : p = 1 \hookrightarrow (2n-1)/3 \\ \text{if } 13 \equiv n \bmod 18 : p = 2 \hookrightarrow (4n-1)/3 \\ \text{if } 17 \equiv n \bmod 18 : p = 1 \hookrightarrow (2n-1)/3 \end{array} \right\}; & & \left\{ \begin{array}{c} 1,19,37,55,\ \ldots \\ 5,23,41,59,\ldots \\ 7,25,43,61,\ldots \\ 11,29,47,65,\ldots \\ 13,31,49,67,\ldots \\ 17,35,53,71,\ldots \end{array} \right\} & \to & \left\{ \begin{array}{r} 1,25,49,73 \\ 13,61 \\ 37 \\ 7,19,31,43,55,67,79,91 \\ 17,41,65,89 \\ 11,23,35,47,59,71,83,95 \end{array} \right\} \equiv R(n) \bmod 96 \end{array}$$

with periodicity expansion factor $\theta_R = 96/18 = 2^4 3^{-1}$
and the rightward heap vector $\overrightarrow{h}_R = [4, 2, 1, 8, 4, 8]$.

For the first row of definition 1, the stated minimum argument $p = 2$ in the formula $(2^p(3n+1) - 1)/3$ results indeed in $U(n) = 4n + 1$ – depicted as light green output expansions in Figure 2. Substitution for n of the arithmetic argument progression $6i + 1$, which corresponds to the congruence class $1 \equiv n \bmod 6$ in the if-part, gives arguments $\{1, 7, 13, 19\ldots\}$. These are turned into $4n+1$ outputs $\{5, 29, 53, 77, \ldots\}$, which is the arithmethic output progression $24i + 1$ with 24 as its periodicity. The lower value $p = 1$ gives the intolerable fractional output progression $12i + 7/3$. Similarly, the stated minimum argument $p = 4$ in the second row gives indeed $16n + 5$ – depicted by the bright green upward output expansions in Figure 2. Given congruence class $5 \equiv n \bmod 6$, the arithmetic argument progression $\{5, 11, 17, \ldots\}$ results in the arithmetic output progression $\{85, 181, 277, \ldots\}$ with $96 = 4 \cdot 24$ as its periodicity. The heap vector $\overrightarrow{h}(a) = [4, 1]$ expresses that 4 successive arguments from $\{1, 7, 13, 19, \ldots\}$ have to be combined with 1 successive argument from $\{5, 11, 17, \ldots\}$ to fill one common output period with the least common output periodicity, which is $LCM(24, 96) = 96 = 2^5 \cdot 3$. This gives five upward output congruence classes $\mathbb{N}^U = \{\{5, 29, 53, 77\}, \{85\}\} \equiv n \bmod 96$, with an obvious density of $\rho\left(\mathbb{N}^U\right) = 5/96$. The upward function expands the periodicity by $\theta_{U1} = 96/6 = 16$. Expressing output congruence classes modulo 96 in modulo 6 reveals that U turns class 1 into class 5 and vice versa. Therefore every second upward iteration amounts to $U^2(n) = 64n+21$, with an output periodicity expansion $\theta_{U2} = 64$ after two upward iterations.

The rightward function is derived similarly, given an argument periodicity of 18 with six argument congruence classes culminating in the rightward heap vector $\overrightarrow{h}_R(a) = [4, 2, 1, 8, 4, 8]$. The two contracting rightward argument congruence classes $\{11, 17\} \equiv n \bmod 18$ are shown in bright red in Figure 2. They motivate picturing trees bending to the right.

The rightward function R_0 gives 27 other, *disjoint*, output congruence classes modulo 96, with an obvious density of $\rho\left(\mathbb{N}^R\right) = 27/96$. The output periodicity expansion for the rightward function amounts to $\theta_R = 96/18 = 2^4 3^{-1}$. Disjoint subsets reflect that branches of a tree do not grow together again. The periodicity expansion factors will be used to assess the geometric density decline at each further iteration.

To define the binary tree $T_{\geq 0}$ in terms of U and R_0, a notation of iterations, inverses and composites is required. Iterates are denoted in superscript, for example as $R_0^2(5) = R_0(R_0(5)) = 17$, $R_0^3(85) = 401$ and $U^5(n) = 21845$, $U^2(17) = 1109$. Their downward and leftward *inverses* are denoted with negative numbers, for example $R_0^{-2}(17) = 5$, $U^{-5}(21845) = 1$. A *root path* connecting the root $\Omega_{\geq 0} = 1$ to a specific number n from its node set $S_{\geq 0}$, denoted as $Q_n(\Omega_{\geq 0})$, is a composite function of upward and rightward iterations, for example $Q_{1813} = U^1 R^1 U^2(1)$. The last function applied comes first in row. Undirected root paths, like $1 - 5 - 13 - 53 - 853 - 3413$ in Figure 2, represent both the *root path* $Q_{3413} = U^3 R^1 U^1 (1)$, and its inverse $U^{-1} R^{-1} U^{-3} (3413) = 1$.

Definition 3. Binary tree $T_{\geq 0}$ is a rooted, directed, infinite binary tree with:

1. an *infinite* node set $S_{\geq 0} \subseteq \mathbb{N}^{UR}$ for $\mathbb{N}^{UR} = \{1, 5\} \equiv n \bmod 6$;
2. an edge set defined by $U : S_{\geq 0} \to S_{\geq 1}$ and $R_0 : S_{\geq 0} \to S_0$, which bifurcate each $S_{\geq 0}$ argument into disjoint subsets $S_{\geq 1}$ and S_0, giving a *directed binary* tree,

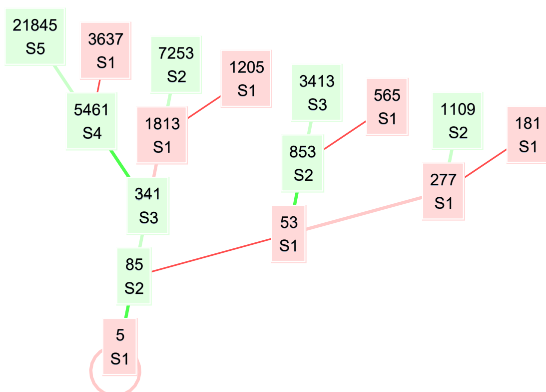
and as implied features:

1. 3 adjacent nodes to each node $n \in S_0$: an upward child $U(n)$, a rightward child $R_0(n)$ and either an upward parent $U^{-1}(n)$ if $n \in S_{\geq 1}$, or a rightward parent $R_0^{-1}(n)$ if $n \in S_0$
2. A *cyclic root* $\Omega_{\geq 0} = 1$ with also 3 adjacent nodes: $U(\Omega_{\geq 0}) = 5$, $R_0(\Omega_{\geq 0}) = 1$, and $R_0^{-1}(\Omega_{\geq 0}) = 1$
3. Commutative iterates of U and R_0, e.g. $U^2(1) = U^{-1} U^3(1) = 85$. This allows for $U^{-j} U^j(n) = n$ excursions to higher nodes $U^j(n)$, which is the basis for definition 4 and lemma 1.
4. Non-commutative, *disjoint*, composites of U and R_0, e.g. $R_0^2 U^2(5) = 151$, $U^2 R_0^2(5) = 1109$.
5. Node subsets defined by iterations of the upward and rightward functions U and R_0:
   (a) Rightward subset $S_o \subseteq \mathbb{N}^R$; $S_0 = 1 \cup \left\{R_0^k(n) \mid n \in S_{\geq 1},\ k \in \mathbb{N}\right\}$; S_0 excludes rightward n with an infinite rightward ancestry (traversals non-trivial cycle, divergent trajectories)
   (b) Upward subset $S_{\geq 1} \subseteq \mathbb{N}^U$; $S_{\geq 1} = \left\{U^j(n) \mid n \in S_o,\ j \in \mathbb{N}\right\}$, which can be further split in even subsets $S_{j \in \{2,4,6,..\}}$ based on iterations of $U^2(n) = 2^{3.2} n + (2^{3.2} - 1)/3$, and odd subsets $S_{j \in \{1,3,5,..\}}$, by applying $U(n)$ first.
   (c) Any even upward subset $S_{j \in \{2,4,6,...\}} = \left\{2^{3j} n + \left(2^{3j} - 1\right)/3 \;\mid\; n \in S_0\right\}$
   (d) Any odd upward subset $S_{j \in \{1,3,5,...\}} = \begin{cases} \left\{2^{3j-1} n + \left(2^{3j-1} - 1\right)/3 \;\mid\; n \in S_0,\ 1 \equiv n \bmod 6\right\} \\ \left\{2^{3j+1} n + \left(2^{3j+1} - 1\right)/3 \;\mid\; n \in S_0,\ 5 \equiv n \bmod 6\right\} \end{cases}$
   (e) Node subset $S_{\geq 0} = \left\{n \mid n \in \mathbb{N}^{UR},\text{ root path } Q_n(\Omega_{\geq 0}) \text{ in tree } T_{\geq 0} \text{ exists,}\right\}$.
   The root path $Q_n(\Omega_{\geq 0})$ of the root $\Omega_{\geq 0}$ towards $n \in S_{\geq 0}$ is a finite sequence of alternating upward and rightward iterations, starting with an upward iteration towards the *upward trunk* $U^{j_1}(\Omega_{\geq 0})$ of the numbers 5, 85, 341, … (see Figure 2). For example, $Q_{1813} = U^1 R^1 U^2(1)$, depicted as 1–5 – 85–113–1813 in Figure 2. The optional nature of further iterations gives the brackets in $Q_n = ((((((R_0^{k_z}) U^{j_z}) \ldots) R_0^{k_2}) U^{j_2}) R_0^{k_1}) U^{j_1}(\Omega_{\geq 0})$.

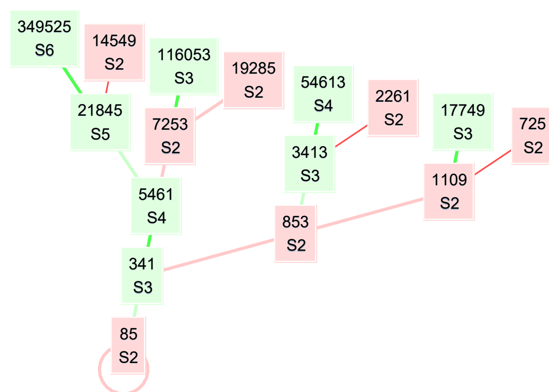
4 An infinite number of infinite departures from the binary tree

Definition 3 partitions the node set $S_{\geq 0}$ in subsets $S_0, S_1, S_2, \ldots$, which will depart successively. A close look at Figures 2 and 3 is the fast way to grasp that after the successive departures of subsets $S_0, S_1, S_2, S_3, \cdots$ each remaining upward generation descends to the positions of their parents. The nodes in the re-erected trees $T_{\geq 1}$, $T_{\geq 2}$, $T_{\geq 3}$, $T_{\geq 4}$ that were occupied in tree $T_{\geq 0}$ by rightward S_0 numbers are successively taken by their S_1 children, S_2 grandchildren, S_3 grandchildren, S_4 great-grandchildren, and so on.

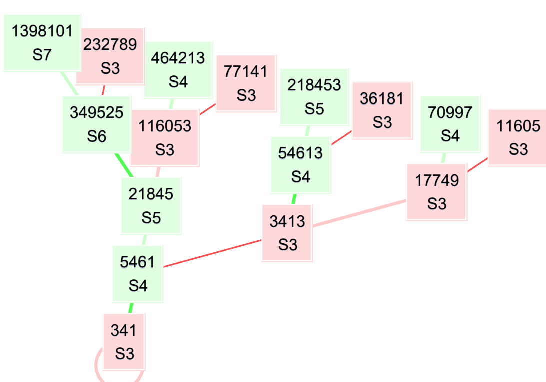
$T_{\geq 1}$ with $\Omega(T_{\geq 1}) = U^1(1) = 5$,
S_0 departed, S_1 (in red) to depart next;
Green upward U, red rightward $R_1 = U^1 R_0^1 U^{-1}$



$T_{\geq 2}$ with $\Omega(T_{\geq 2}) = U^2(1) = 85$,
S_1 departed, S_2 (in red) to depart next;
Green upward U, red rightward $R_2 = U^2 R_0^1 U^{-2}$



$T_{\geq 3}$ with $\Omega(T_{\geq 3}) = U^3(1) = 341$,
S_2 departed, S_3 (in red) to depart next;
Green upward U, red rightward $R_3 = U^3 R_0^1 U^{-3}$



$T_{\geq 4}$ with $\Omega(T_{\geq 4}) = U^4(1) = 5461$,
S_3 departed, S_4 (in red) to depart next;
Green upward U, red rightward $R_4 = U^4 R_0^1 U^{-4}$

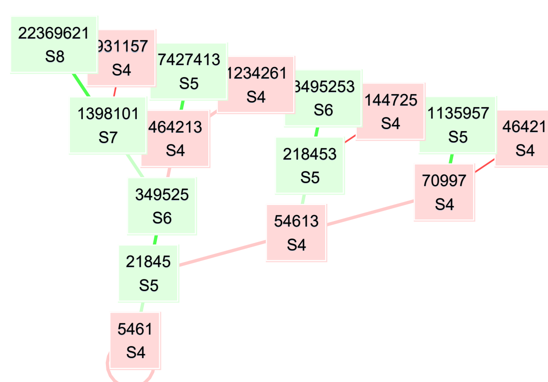


Figure 3: *Reconnected trees $T_{\geq 1}, T_{\geq 2}, T_{\geq 3}, T_{\geq 4}$ after the departures of subsets S_0, S_1, S_2, S_3*

The numbers on red orbits get further apart in each successive tree, which indicates their ever lower density due to upward periodicity expansions. As can be seen from Figure 3, the red coloured reconnections in any tree $T_{\geq j}$ are generated by the rightward reconnection function $R_j(n) = U^j R_0^1 U^{-j}(n)$, which expresses how the underlying rightward iteration R_0^1 is expanded by j iterations of U. For $j = 0$, $R_j(n)$ collapses to $R_j(n) = U^0 R_0 U^0(n) = R_0(n)$.

Definition 4: Rightward reconnection function R_j in tree $T_{\geq j}$

$$R_j: \; S_{\geq j} \to S_j; \quad R_j(n) = U^j R_0 U^{-j}(n)$$

Let's now come to theorem 1 that each root path in any remaining tree $T_{\geq j}$ simply mirrors a root path in the basic binary tree $T_{\geq 0}$, such that proving a root path in $T_{\geq j}$ suffices to prove that a corresponding root path existed in $T_{\geq 0}$. A close look at Figures 2 and 3 is helpful again. Let's imagine being the root host $\Omega_{\geq 2} = 85$ of Hilbert hotel $T_{\geq 2}$, depicted in the up-right panel of Figure 3. $\Omega_{\geq 2}$ wants to check, without having access to prior hotels $T_{\geq 0}$ and $T_{\geq 0}$, whether guest $n = 1813$, who already departed with subset S_1, had a valid *root path* in tree $T_{\geq 0}$. Since all rooms in hotel $T_{\geq 2}$ were inherited from grandparents in hotel $T_{\geq 0}$, it is helpful to identify $\Omega_{\geq 0} = 1 = U^{-j}(\Omega_{\geq 2})$ as the grandparent of $\Omega_{\geq 2}$, and $n = 1813$ with an unverified root path $Q_{1813}(\Omega_{\geq 0})$ in tree $T_{\geq 0}$ as the grandparent $U^{-2}(116053) = 1813$ of $T_{\geq 2}$ guest 116053. Finding in hotel $T_{\geq 2}$ the root path $85 - 341 - 5461 - 7253 - 116053$, denoted as $Q_{116053}(\Omega_{\geq 2}) = U^1 R_2^1 U^2(85)$, ensures that $\Omega_{\geq 0} = U^{-2}(85) = 1$ could have verified the root path $1 - 5 - 85 - 113 - 1813$ in Figure 2, denoted as $Q_{1813}(\Omega_{\geq 0}) = U^1 R_0^1 U^2(1)$ towards 1813. Similarly hosts $\Omega_{\geq 3} = 341$ and $\Omega_{\geq 3} = 5461$ of hotels $T_{\geq 3}$ and $T_{\geq 4}$ can verify that 1813 had a root path in $T_{\geq 0}$ by verifying in their own trees the root paths to the unique upward descendants 464213 and 7427413 of 1813, $341-5461-21845-116053-464213$ and $5461 - 21845 - 349525 - 464213 - 7427413$.

Theorem 1: A root path $Q_n(\Omega_{\geq j})$ exists from the root $\Omega_{\geq j} = U^j(\Omega_{\geq 0})$ in the node subset $S_{\geq j}$ of tree $T_{\geq j}$ to node n in tree $T_{\geq j}$, if and only if in the basic binary tree $T_{\geq 0}$ a root path $Q_m(\Omega_{\geq 0})$ exists from the root $\Omega_{\geq 0}$ to the j'th downward ancestor $m = U^{-j}(n)$ of n.

Proof. From the point of view of tree $T_{\geq 0}$, with a root path $U^j Q_m(\Omega_{\geq 0})$ towards the upward child $n = U^j(m)$ of node m, the root path $Q_n\left(\Omega_{\geq j}\right)$ in tree $T_{\geq j}$ is a higher located parallel path to which definition 4 allows excursions, first of j steps up, and next of j steps down. Let's *prove* this for a root path to n in tree $T_{\geq j}$ with one rightward reconnection R_j^k, thus for $Q_n\left(\Omega_{\geq j}\right) = R_j^k U^i(\Omega_{\geq j})$, which should mirror $Q_m(\Omega_{\geq 0}) = R_0^k U^i(\Omega_{\geq 0})$ towards m in tree $T_{\geq 0}$. Substituting $\Omega_{\geq j} = U^j(\Omega_{\geq 0})$ gives path $Q_n\left(\Omega_{\geq j}\right) = R_j^k U^i(U^j(\Omega_{\geq 0}))$ to n which should mirror $Q_n(\Omega_{\geq 0}) = U^j(R_0^k U^i(\Omega_{\geq 0}))$ in tree $T_{\geq 0}$. Applying definition 4 next gives $Q_n\left(\Omega_{\geq j}\right) = U^j R_0^k U^{-j}(U^i(U^j(\Omega_{\geq 0})))$. Cancelling the excursion to $\Omega_{\geq j}$ gives $Q_n\left(\Omega_{\geq j}\right) = U^j R_0^k U^{-j+i+j}(\Omega_{\geq 0}) = U^j(R_0^k U^i(\Omega_{\geq 0})) = U^j Q_m(\Omega_{\geq 0})$, which is indeed the root path to n starting with root path $Q_m(\Omega_{\geq 0})$ to m in tree $T_{\geq j}$. This is not surprising. All infinite binary trees are *isomorph* to each other *(8)*, since there is a one-to-one correspondence between their node sets which preserves adjacency, also the adjacency of nodes in corresponding root paths.

5 Density of departed subsets

Let's now prove that the density of remaining subsets $S_{\geq j}$ on which to prove a root path comes arbitrarily close to zero if j is increased to arbitrarily high values, by proving that the density of already departed subsets comes arbitrarily close to 1.

Theorem 2: Density departed subsets from T_C

$$\begin{aligned}\rho(T_C) &= \rho(S_{-1}) + \rho(S_0) + \rho(S_{1,3,5,\ldots}) + \rho(S_{2,4,6,\ldots}) \\ &= 4/6 + 27/96 + 1/21 + 1/224 = 1\end{aligned}$$

The rows in Table 1 show the steps in the proof for each of the four subsets from Theorem 2 in its columns. For its first column regarding the density of the preliminarily departed subset S_{-1} of numbers divisible by 2 and/or 3 the basic notation was already introduced: a periodicity of $\nu = 6$, a cardinality per period of $\#(S_{-1} \mid \nu) = 4$, and a density of $\rho(S_{-1}) = 4/6$.

Proofs for the three remaining subsets can best start from the proof with the least required proof steps, which is the proof that $\rho(S_{2,4,6,\ldots}) = 1/224$. It starts from definition 3 of upward even subsets $S_{j\in\{2,4,6,\ldots\}} = \left\{2^{3j}n + \left(2^{3j}-1\right)/3 \mid n\in S_0\right\}$ generated by rightward S_0 ancestor arguments. Their density is $\rho(S_0) = 27/96$ by the proof below. Their cardinality per period $\nu_0 = 96 = 2^5 3$ amounts to $\#(S_0|\nu_0) = 27$. Each $n \in S_0$ has exactly one upward descendant $U^j(n)$ in upward subset S_j. The upward expansion factor $\theta_{U2} = 64 = 2^{2\cdot 3}$ expands the periodicity of arguments from $2^5 3$ to $2^{3j+5}3$. Therefore the density of the even subset S_j amounts to $\rho\left(S_j\right) = 27/(2^{3j+5}3)$, with as first even term $a = \rho(S_2) = 27/(2^{3\cdot 2+5}3) = 27/6144$. The density decay rate amounts to $r = 1/\theta_{U2} = 1/64$. The familiar sum of an infinite geometric series, $s = a/(1-r)$ gives as density indeed $\rho(S_{2,4,6,\ldots}) = (27/6144)/(1-1/64) = 1/224$.

The upward periodicity expansion $\theta_{U2} = 2^{3\cdot 2}$ after each second iteration is preceded in the assessment of the density of odd upward subsets $\rho(S_{1,3,5,\ldots}) = 1/21$ by one upward iteration to assess the density of the first term $a = \rho(S_1)$. The upward heap vector $\overrightarrow{h}_U = [4,1]$ from definition 1 combines 4 successive arguments $1 \equiv n \bmod 6$ with 1 argument $5 \equiv n \bmod 6$ to fill one, $\theta_{U1} = 16$ times expanded, output period. The argument cardinality vector $\overrightarrow{\#}(S \mid \nu_0) = [15,12]$ indicates that 15 out of the 27 rightward congruence classes $\{1,7,13,19,25,31,37,43,49,55,61,67,73,79,91\} \equiv$ mod 96 belong to $1 \equiv n \bmod 6$. The output periodicity, given by $\nu_1 = 96\,\theta_{U1}$, amounts to $\nu_1 = 6144 = 2^{11}3$. This gives a total cardinality $\#(S_1|\nu)$ of $4\cdot 15 + 1\cdot 12 = 72$, and a density $a = \rho(S_1) = 72/6144$. The density of all odd upward sets is therefore indeed $\rho(S_{1,3,5,\ldots}) = a/(1-r) = (72/6144)/(1-1/64) = 1/21$.

In calculating the density $\rho(S_0) = 27/96$ of rightward numbers generated by rightward iterations $R_0^k : S_{\geq 1} \rightarrow S_{0k}$ of the rightward function, starting from upward arguments $n \in S_{\geq 1}$, the first iteration is special because upward arguments are turned into rightward numbers in subset S_{01}, whose density will give the first term of a geometric series $a = \rho(S_{01})$. The least common multiple periodicity LCM$(96,18) = 288 = 2^5 3^2$ guarantees that each argument period mod 288 comprises 16 rightward argument classes mod 18 required for rightward iterations, with $3\cdot 5$ upward arguments. Their distribution over congruence classes $\overrightarrow{\boldsymbol{a}} = [1,5,7,11,13,17]$ mod 18, $[\{181\},\{5,77,149,221\},\{277\},\{29,101,173,245\},\{85\},\{53,125,197,269\}]$ mod 288, gives the argument cardinality vector $\overrightarrow{\#}(S|\nu_{0\rightarrow 1}) = [1,4,1,4,1,4]$.

Table 1: Densities, periodicities and cardinalities per period of arguments and outputs

Departed node subsets S	S_{-1}	S_0	$S_{j\in\{1,3,5,\dots\}}$	$S_{j\in\{2,4,6,,\dots\}}$
Generating function	$\mathbb{N} \to S_{-1}$; divisible by 2 or 3	$R_0^k: S_{\geq 1} \to S_{0k}$ $k = 1,2,3,\dots$	$U^j: S_0 \to S_j$ odd $j = 1,3,5,\dots$	$U^j: S_0 \to S_j$ even $j = 2,4,6,\dots$
Argument subset	$\mathbb{N}$	$S_{\geq 1}$	S_0	S_0
periodicity ν_0	6	$2^5 3^{k+1}, \nu_{0\to 1} = 288$	$2^5 3 = 96$	$2^5 3 = 96$
cardinality $\vec{\#}(S\|\nu_0)$	·	[1,4,1,4,1,4]	[15,12]	∘
total $\#(S\|\nu_0)$	4	15	$3^3 = 27$	$3^3 = 27$
Special first iteration	·	$S_{01} = \{R_0^1(n) \mid n \in S_{\geq 1}\}$	S_1	∘
period expansion θ	·	θ_R	θ_{U1}	∘
periodicity ν_1	·	$2^{4\cdot 1+5}3 = 1536$	$2^{3\cdot 1+6}3 = 1536$	∘
cardinality $\vec{\#}(S\|\nu_1)$	·	$\vec{\#}(S_{01}\|\nu_{01})\,\vec{h}_R =$ [4,8,1,32,4,32]	$\vec{\#}(S_1\|\nu_1)\,\vec{h}_U =$ [60,12]	∘
total $\#(S\|\nu_1)$	·	$3^4 = 81$	$2^3 3^2 = 72$	∘
$a = \rho$(output first iteration)	·	$81/1536$	$72/1536$	$27/(2^{3\cdot 2+5}3) = 27/6144$
Output subset (further) iterations	·	$S_{0k,k>1}$	$S_{i\in\{3,5,\dots\}}$	$S_{j\in\{2,4,6,,\dots\}}$
period expansion θ	·	θ_R	θ_{U2}	θ_{U2}
periodicity ν_i	·	$\nu_{k-1}\theta_R = 2^{4k+5}3$	$\nu_{j-2}\theta_{U2} = 2^{3j+6}3$	$\nu_{j-2}\theta_{U2} = 2^{3j+5}3$
cardinality $\vec{\#}(S\|\nu_i)$	·	$\left(\boldsymbol{T}' \cdot \vec{\#}(S\|\nu_{i-1})\right)\vec{h}_R$	∘	∘
total $\#(S\|\nu_i)$	·	$81 \cdot 13^{k-1}$	72	27
density $\rho(S_i)$	·	$\rho(S_{0k}) = (81 \cdot 13^{k-1})/(2^{4k+5}3)$	$\rho(S_i) = 72/(2^{3j+6}3)$	$\rho(S_j) = 27/(2^{3j+5}3)$
$r = \rho(S_i)/\rho(S_{\text{prior\_}i})$	·	$13/\theta_R = 13/16$	$1/\theta_{U2} = 1/64$	$1/\theta_{U2} = 1/64$
Departed densities $\rho(S) = a/(1-r)$	$\rho(S_{-1}) = 4/6$	$\rho(S_0) = 27/96$	$\rho(S_{1,3,5,\dots}) = 1/21$	$\rho(S_{2,4,6,\dots}) = 1/224$
cumulative	4/6	91/96	223/224	1

Table Note. Missing cells: ∘ not required, · not applicable. Additional notation in table cells: from definitions 1 and 2 the periodicity expansion factors $\theta_R = 96/18 = 2^4 3^{-1}$, $\theta_{U1} = 96/6 = 2^4$ and $\theta_{U2} = 64 = 2^6$, as well as the heap vectors $\vec{h}_U = [4,1]$ and $\vec{h}_R = [4,2,1,8,4,8]$; $\boldsymbol{T}'$ is the *transposed* rightward transformation matrix with 6 rows, alternatingly $[1,1,1,1,0,0]$ and $[0,0,0,0,1,1]$ involved in the matrix multiplication · in $\left(\boldsymbol{T}' \cdot \vec{\#}(S \mid \nu_{i-1})\right)\vec{h}_R$.

Multiplication of the argument cardinality vector with the rightward heap vector $\overrightarrow{h} = [4,2,1,8,4,8]$ to map argument periods to their least common output periodicity gives as output cardinality vector $[4,8,1,32,4,32]$ with a total cardinality of $81 = 3^4$, resulting in the required density $a = \rho(S_{01}) = 81/(288\ \theta_R) = 81/1536$.

The transformation matrix T indicates how the rightward function $R(n)$ distributes its outputs to congruence classes mod 18 that serve as arguments for the next rightward iteration. $R^2(n)$ has a tripled periodicity $3 \cdot 18 = 54$. Each argument out of the six congruence classes $\{1,5,7,11,13,17\}$ mod 18 produces in each tripled period periodically the same output congruence classes modulo 18, which are successively $\{\{1,7,13\},\{13,7,1\},\{1,7,13\},\{7,1,13\},\{17,5,11\},\{11,5,17\}\}$ mod 18. For example, for $a = 13$ the arithmetic argument progression $\{13,31,49,\ 67,85,103,\ 121,139,57,\ \ldots\}$ produces the output progression $\{17,41,65,\ \ 89,113,137,\ \ 161,185,209,\ \ \ldots\}$, in which each triplet reduces to the congruence classes $\{17,5,11\} \equiv n$ mod 18. Neglecting order then gives the 6 by 6 transformation matrix T with as first 4 rows $[1,0,1,0,1,0]$ and as its last two rows $[0,1,0,1,0,1]$. Its transpose T' indicates the cardinality of arguments of each congruence class mod 18 to be entered in the next rightward iteration, given the argument cardinality vector at the previous rightward iteration. The rightward heap vector is again required to fill each rightward output period. The thus compiled formula $\left(\boldsymbol{T}' \cdot \overrightarrow{\#}\,(S \mid \nu_{i-1})\right)\overrightarrow{h}_R$ results in a rightward cardinality expansion factor of 13 at each further rightward iteration. The sum of the powers of 2 in definition 2 of the rightward function was also 13. At each further rightward iteration, the argument period is tripled again, e.g. from 54 to 162, which gives an output periodicity expansion of $3\theta_R = 3 \cdot 2^4 3^{-1} = 16$, and a rightward density decay factor of $r = 13/16$, giving indeed $\rho(S_0) = a/(1-r) = (81/1536)/(1-13/16) = 27/96$.

The four density calculations imply that the cumulative density of departed node subsets from the Collatz tree comes arbitrary close to 100% of the natural numbers. In combination with theorem 1, maintaining that all departed numbers had a root path provided they would have a root path in the set of remaining subsets, whose density is now proven to come arbitrarily close to 0%, this proves that $N(T_C) = \mathbb{N}$.

6 Discussion

Proving the Collatz conjecture by considering a Hilbert hotel with the archictecture of a Collatz tree reveals new links between graph theory and number theory, as hoped for by its originator (5). To our knowledge it's the first Hilbert hotel that is construed as a number tree rather than as a number line.

A momentous feature of the binary tree is its one-to-one mapping of unique numbers *on* its nodes to unique binary root paths of either upward or rightward steps from its root *to* each of its nodes. For example, $35 = 2^0 3^0 5^1 7^1 \leftrightarrow \langle\, 0,0,1,1\rangle$ if 35 is factorized, or $35 = 1 \xrightarrow{1} 5 \xrightarrow{0} 13 \xrightarrow{1} 53 \xrightarrow{0} 35 \leftrightarrow \langle 1,0,1,0\rangle$ if the binary Collatz path from the root to 35 is taken. The proof of the Collatz conjecture may enhance recently explored applications of the Collatz tree to random number generation, encryption, and watermarking.

Ancillary Materials. *ElaborationProofsCollatzTree.pdf* gives an account of the notation used with further examples and elaborations, especially of various tree plots, and of the periodicities and cardinalities that underlie the density calculations for Lemma 4. *Math12NotebookCollatzTree.nb*, with additionally its static pdf, is the Mathematica 12 version of the Mathematica notebook underlying the paper.

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Works 11-66 from the references below are cited in the ancillary materials. [1] [2] [3] [4] [5] [6] [7] [8] [9] [10] [11] [12] [13] [14] [15] [16] [17] [18] [19] [20] [21] [22] [23] [24] [25] [26] [27] [28] [29] [30] [31] [32] [33] [34] [35] [36] [37] [38] [39] [40] [41] [42] [43] [44] [45] [46] [47] [48] [49] [50] [51] [52] [53] [54] [55] [56] [57] [58] [59] [60] [61] [62] [63] [64] [65] [66] [67]

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